Part 1 A. Analysis and Simulation of Power Electronic Circuits (Continued)

Dynamics of the average- state-space averaging

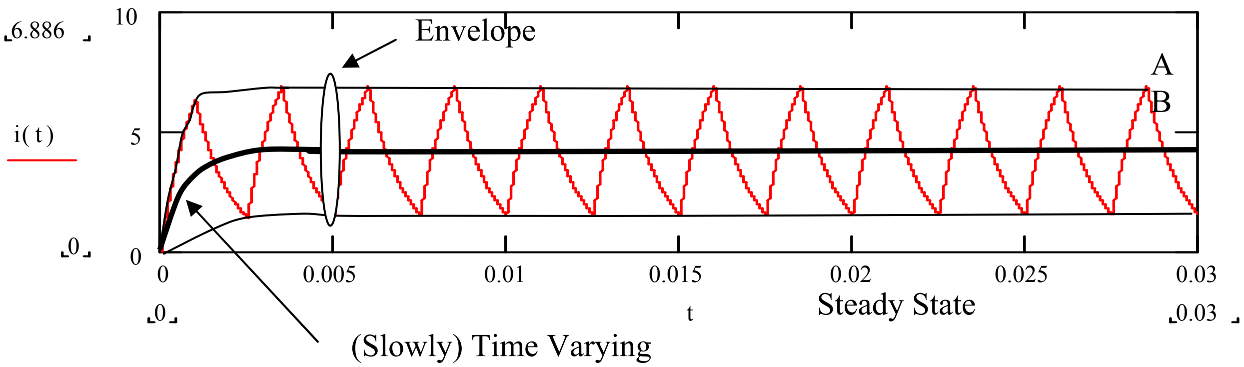
In the previous lectures we developed a complete analytical solution for periodically switched circuits, specifically, the chopper circuit. Such solutions are at best difficult to obtain. Next we found that many features of the solution can be very effectively derived using the so-called averaging principle. The most important application of the averaging principle is in deriving formulas for average voltages and currents. Simple approximations can also be used to estimate steady-state ripple.

We will now be interested in the dynamic response of these converters.

Preliminaries

In order to define what we mean by "dynamic response" consider the current shown below for the chopper circuit. At time t=0 the chopper circuit is turned on with duty ratio set to 0.4. In each cycle the current increases and decays exponential governed by the RL time constant. The overall current as, illustrated by the envelope, also appears to increase exponentially until steady state is established. In steady state we have described the current by its average value plus the ripple.

Dynamics describes the time behavior of the buildup of current or the general shape of the current envelope.



(Slowly) Time Varying average

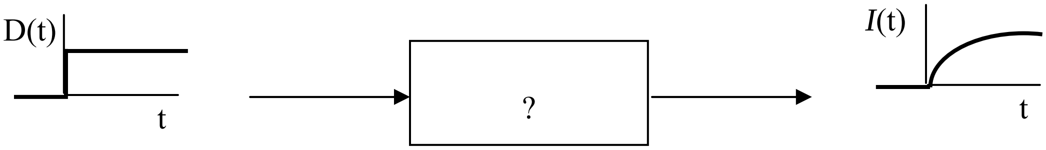
Conventional Average in periodic steady state

The dynamics is convenienty describe by the time varying average current

$I(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} i(t)dt$. Note that $I(t)|_{t\to\infty}$ = Iaverage

This definition is most useful when the average varies slowly as compared to the switching period.

Now think of the above response as follows. At t=0- the converter was energized but duty ratio D was set to zero. At t=o a step change of 0.4 was made to the duty ration. We are interested in the response of I(t) as it gets to steady state.



In general the relationship is both dynamic and nonlinear. It is the starting point in the design of control systems to regulate the desired output.

State Space Averaging

Among many approaches to deriving the equations for converter dynamics, State Space Averaging is particularly useful. The method is summarized below.

Assume that the switching frequency is much higher than the natural frequencies of the converter.

A. In each switching period T, as switches open and close, the circuit goes through K 'circuit states' or 'circuit modes[1]', as described by circuit topology
B. Circuit state k last for time t_k. Note

$$\sum_{k=1}^{k} t_k = T$$

Where T = 1/f is the switching frequency
C. For each circuit state k, k=1,2…K the converter is described by an appropriate state-variable model. Assuming linear circuit components (R, L, C…) the form of the state equation in circuit mode k is

$$d\,\underline{x}(t)/dt = A_k\,\underline{x}(t) + B_k\,\underline{u}(t)$$
$$\underline{y}(t) = C_k\,\underline{x}(t) + D_k\,\underline{u}(t)$$

Here, $\underline{x}(t)$ represents the state vector, $\underline{u}(t)$ the inputs, and $\underline{y}(t)$ the output vector

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[1] The circuit mode should not be confused with DCM and CCM.

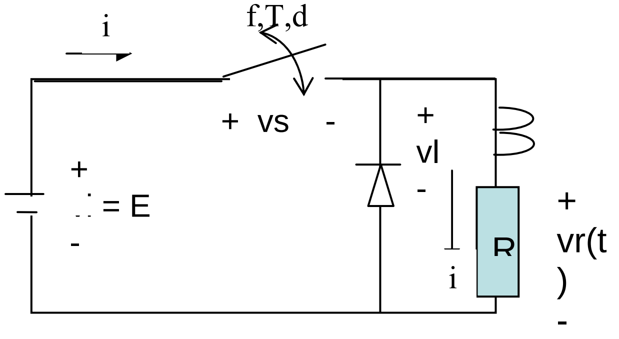
The state space averaging principle states that the slowly time-varying averages *X*(t) and *Y*(t) satisfy the state equation

$$d\underline{X}(t)/dt = A\,\underline{X}(t) + B\,\underline{u}(t)$$
$$\underline{Y}(t) = C\,\underline{X}(t) + D\,\underline{u}(t)$$

Where

$$A = \frac{1}{T}\sum_{k=1}^{K} t_k\, A_k \qquad B = \frac{1}{T}\sum_{k=1}^{K} t_k\, B_k \qquad C = \frac{1}{T}\sum_{k=1}^{K} t_k\, C_k \qquad D = \frac{1}{T}\sum_{k=1}^{K} t_k\, D_k$$

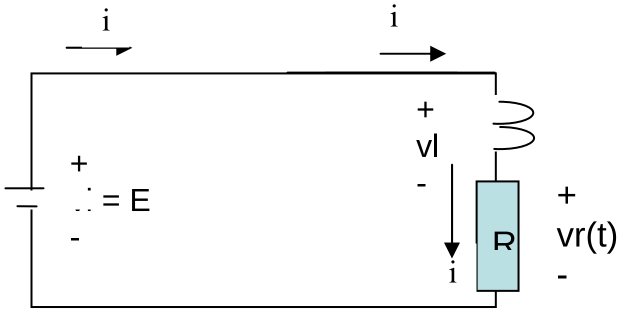
Example 1: Chopper with inductive filter and free wheeling diode



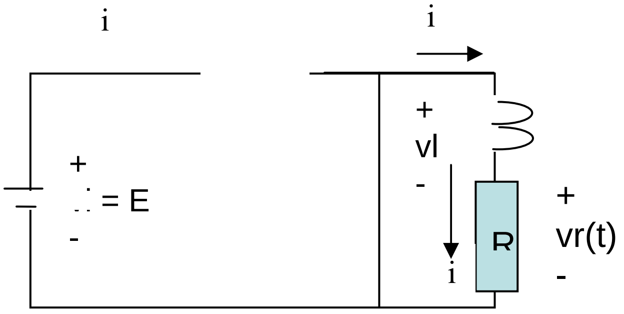
Recall that the chopper below converts the dc voltage into a dc current whose dc or average value can be controlled by the duty cycle. We wish to determine the dynamics of average load current

A. Circuit states

For CCM the chopper has the two circuit states shown below



State 1 (k=1) Switch on



State 1 (k=2) Switch off

States, Outputs and Inputs:

The state vector is a singleton corresponding to inductor current i(t). The ouput of interest is also the inductor current i(t). The inputs can be considered to be the source voltage E and duty ratio D(t)

Thus,

$\underline{x}(t)=i(t)$ $\qquad\qquad$ $\underline{y}(t)=i(t)$ $\qquad\qquad$ $u(t)=\begin{bmatrix} E \\ D(t) \end{bmatrix}$

State equations:

Circuit state 1

Since $E= Ri(t) + L\, di(t)/dt$

$$\begin{aligned} d\,\underline{x}(t)/dt &= [-R/L]\; x(t) + [\;1/L \;\; 0\;]\; \underline{u}(t) \\ \underline{y}(t) &= [1]\; \underline{x}(t) \quad + [0]\; \underline{u}(t) \end{aligned}$$

Thus, $A_1 =[-R/L]$ $\quad$ $B_1 =[\;1/L \;\; 0\;]$ $\quad$ $C_1=[0]$ $\qquad$ $D_1 =[-R/L]$

Circuit state 2

Since $0= Ri(t) + L\, di(t)/dt$

$$\begin{aligned} d\,\underline{x}(t)/dt &= [-R/L]\; x(t) + [\;0 \;\;\; 0\;]\; \underline{u}(t) \\ \underline{y}(t) &= [1]\; \underline{x}(t) \quad + [0]\; \underline{u}(t) \end{aligned}$$

Thus, $A_2 =[-R/L]$ $\quad$ $B_2 =[\;0 \;\; 0\;]$ $\quad$ $C_2=[0]$ $\qquad$ $D_2 =[-R/L]$

State equation for average current:

Using the state space averaging principle,

$$\begin{aligned} d\underline{X}(t)/dt &= A\,\underline{X}(t) + B\,\underline{u}(t) \\ \underline{X}(t) &= C\,\underline{X}(t) + D\,\underline{u}(t) \end{aligned}$$

Where,

$A=[\;\{DT(-R/L)+(1-D)T(-R/L)\}/T] = [\;-R/L\;]$
$B= [\;\{D\,T(1/L) + (1-D)T(0)\}/T \quad \{D(0)T+(1-D)T(0)\}/T] = [\quad D/L \;\; 0]$

Similarly, $C=[1]$ $\qquad$ $D=[0]$

Summarizing, with X(t) replaced by the average current $I(t)$ for clarity

$$d\,I(t)/dt = (-R/L)\,I(t) + D(t)\,E(t)/L$$

Application of the state-space averaged model

A. Steady state solution

The average model is

$$\begin{aligned} d\underline{X}(t)/dt &= A\,\underline{X}(t) + B\,\underline{u}(t) \\ \underline{Y}(t) &= C\,\underline{X}(t) + D\,\underline{u}(t) \end{aligned}$$

In the steady stae the averages become constant, i.e., $d\underline{X}(t)/dt = \underline{0}$.

Thus, the steady state value of $\underline{X}(t)$ is

$$\underline{X}^{ss}(t) = -A^{-1}B\,u(t)$$

Example: For the chopper the in the steady state D(t)= D and E(t) = E are constant. The average steady-state current us

$$I_{avg} = -A^{-1}B\,u(t) = (-R/L)^{-1}\,D\,E/L = DE/R \;!$$

B. Dynamics

We are interested in the averaged state-space model because we are interest in the dynamics of slowly-varying averages. As indicated previously, we want to know the time variation of average voltage when we make a change to the duty cycle, for example.

The averaged state space equation

$$d\underline{X}(t)/dt = A\,\underline{X}(t) + B\,\underline{u}(t)$$

is actually nonlinear. For the chopper the control inputs of interest are duty ratio D(t) and input voltage E(t). The averaged equation

$$d\,I(t)/dt = (-R/L)\,I(t) + D(t)\,E(t)/L$$

is nonlinear in control input because it involves the product D(t) E(t).

Although nonlinear control methods can be used in designing control systems, we often use a ‘linearized’ model for small variations around a steady state or equilibrium point of interest.

This small signal model is developed as follows.

Let $\underline{X}^{ss}$, $\underline{U}^{ss}$ and $\underline{Y}^{ss}$ represent the state, input and output for a steady state operating point for a converter. Suppose now that the input $\underline{U}^{ss}$ is perturbed by a small amount $\Delta\underline{U}(t)$. As a result the state will also change by the amount $\Delta\underline{X}(t)$, and the output will change by $\Delta\underline{Y}(t)$. If the matrices A and B involve the control variables then these matrices also change by ΔA and ΔB, respectively. These changes are described by the dynamic equation

$$\begin{aligned} d(\underline{X}^{ss} + \Delta\underline{X}(t))/dt &= (A+\Delta A)(\underline{X}^{ss} + \Delta\underline{X}(t)) + (B+\Delta B)(\underline{U}^{ss} + \Delta\underline{U}(t)) \\ \underline{Y}^{ss}(t) + \Delta\underline{Y}(t) &= (C+\Delta C)(\underline{X}^{ss} + \Delta\underline{X}(t)) + (D+\Delta D)(\underline{U}^{ss} + \Delta\underline{U}(t)) \end{aligned}$$

Note that for dc-dc converters $\underline{X}^{ss}$. Neglecting second order terms such as $\Delta A\,\Delta\underline{X}(t)$, we have the approximate model

$$\begin{aligned} d(\Delta\underline{X}(t))/dt &= A\underline{X}^{ss} + B\underline{U}^{ss} + \Delta A\,\underline{X}^{ss} + \Delta B\underline{U}^{ss} + A\,\Delta\underline{X}(t) + B\Delta\underline{U}(t) \\ \Delta\underline{Y}(t) &= C\underline{X}^{ss} + D\underline{U}^{ss} + \Delta D\,\underline{U}^{ss} + \Delta C\,\underline{X}^{ss} + D\,\Delta\underline{U}(t) + C\,\Delta\underline{X}(t) \end{aligned}$$

In the steady state

$$\begin{aligned} 0 &= A\,\underline{X}^{ss} + B\,\underline{U}^{ss} \\ \underline{Y}^{ss} &= C\,\underline{X}^{ss} + D\,\underline{U}^{ss} \end{aligned}$$

Substituting, he equations for perturbed quantities are,

$$\begin{aligned} d\Delta\underline{X}(t)/dt &= A\,\Delta\underline{X}(t) + B\Delta\underline{U}(t) + \Delta A\,\underline{X}^{ss} + \Delta B\underline{U}^{ss} \\ \Delta\underline{Y}(t) &= C\,\Delta\underline{X}(t) + D\,\Delta\underline{U}(t) + \Delta D\,\underline{U}^{ss} + \Delta C\,\underline{X}^{ss} \end{aligned}$$

The above equations represent the state model for small perturbations

Example: Small signal control model for chopper

The averaged equation,

$$d\,I(t)/dt = (-R/L)\,I(t) + D(t)\,E(t)/L,$$

Denoting the steady state duty and input voltage by D^{ss} and E^{ss} , the linearized form is

$$d\Delta\,I(t)/dt = (-R/L)\,\Delta I(t) + D^{ss}\,\Delta E(t)/L + E^{ss}\,\Delta D(t)/L$$

Application of linearized model.

One important application of the linearized model is in deriving the transfer function between control input and the controlled output, for example, duty cycle and output dc voltage.

Given the system

$$d\underline{X}(t)/dt = A\,\underline{X}(t) + B\,\underline{U}(t) \quad \underline{Y}(t) = C\,\underline{X}(t) + D\,\underline{U}(t)$$

Laplace transformation with zero initial conditions yields

$$s\underline{X}(s) = A\,\underline{X}(s) + B\,\underline{u}(s) \qquad \underline{Y}(s) = C\,\underline{X}(s) + D\underline{U}(s)$$

Now we can write the input –putput relation in the Laplace domain as

$$\underline{Y}(s) = \{C[sI-A]^{-1}B + D\}\underline{U}(s)$$

where I is the identity matrix.

The desired transfer function can be obtained from the above relationship.

Example: Chopper transfer function

The averaged small-signal model is,

$$d\Delta I(t)/dt = (-R/L)\,\Delta I(t) + D^{ss}\,\Delta E(t)/L + E^{ss}\,\Delta D(t)/L$$

Thus $\underline{X}(t) = \Delta I(t)$ and $U(t) = \begin{bmatrix} \Delta E(t) \\ \Delta D(t) \end{bmatrix}$

$$A = [-R/L] \qquad B = [\,D^{ss}/L \quad E^{ss}/L\,]$$

The output desired is the current, thus the output equation is

$$\Delta I(t) = I\,\Delta I(t)$$

Thus C=I, the identity matrix and the matrix D = 0

Thus we have,

$$[sI-A]^{-1} = [1/(s+R/L)]$$

and,

$$\{C[sI-A]^{-1}B + D\} = [1/(s+R/L)]\,[\,D^{ss}/L \quad E^{ss}/L]$$

Thus,

$$\Delta I(s) = \{D^{ss}/(R+Ls)\}\,\Delta E(s) \;+\; \{E^{ss}/(R+Ls)\}\,\Delta D(s)$$

Now suppose E is constant, so $\Delta E(s) = 0$.

$$\Delta I(s) = \{E^{ss}/(R+Ls)\}\,\Delta D(s)$$

Thus the transfer function H_{DI} between duty ratio (control input) and chopper average current (controlled output) is

$$H_{DI} = \Delta I(s) / \Delta D(s) = \{E^{ss}/(R+Ls)\}$$

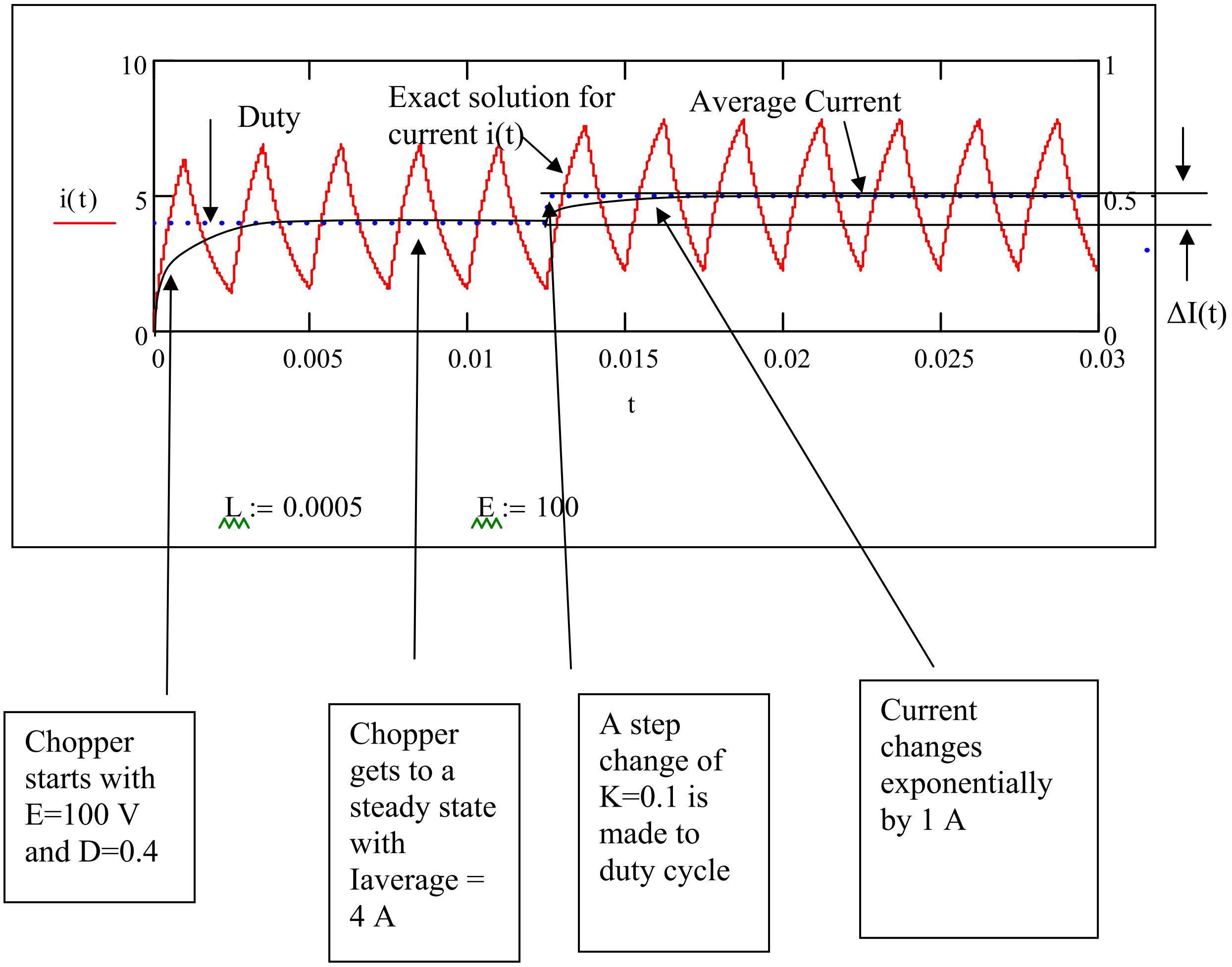
Let's make a step change in duty, $\Delta D(s) = K/s$; then

$$\Delta I(s) = \{E^{ss}/(R+Ls)\}\,\Delta D(s) = K\,E^{ss}/\{s(R+Ls)\}$$

Inverting, we obtain

$$\Delta I(t) = (K\,E^{ss}/R)(1-e^{-tR/L})$$

The output current build up exponentially with time constant $\tau = L/R$; the steady state change in current is $(K\,E^{ss}/R)$ which is directly proportional to the duty cycle change K as illustrated below. (Recall the steady state current in the chopper is Iavg = DE/R)



Chopper starts with E=100 V and D=0.4

Chopper gets to a steady state with Iaverage = 4 A

A step change of K=0.1 is made to duty cycle

Current changes exponentially by 1 A